Behavior of Satellite Objects in the Earth-Moon Lagrange Points

Matthew Beckler

Lab Partners:
Josh Erdman, Jesse Comb

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1 Introduction

In the 18^{th} and 19^{th} centuries, many mathematicians worked extensively on what is known as the '*Restricted Three-Body Problem,*' which is a simplification of the general three-body problem. The aim of the three-body problem is to describe the motions of three planetary bodies, but is impossible to solve analytically. The restriction assumes that the mass of one of the bodies is negligible, such as the mass of a satelite compared to the masses of the earth and moon. In 1777, an Italian mathematician named Joseph Lagrange hypothesized how that restricted third body would interact with the other two bodies. He discovered distinct points in the orbit where the third body would orbit at a relatively stationary position compared to the other two. These points were later named Lagrange points in his honor.

In this lab, we created a computer simulation to observe and quantitize the behavior of objects placed at these Lagrange points. We investigated the overall stability of the five Lagrange points, and also the effect of the ratio of the masses of the larger two bodies on the stability of the orbits.

2 Predictions

There are five Lagrange points, enumerated L1-L5:

- L1 lies between the earth and the moon
- L2 lies outside the moon's orbit directly behind it
- L3 lies on the other side of the earth from the moon
- L4 is at the same radius as the moon, but 60° ahead of the moon
- L5 is the same as L4, but 60° behind the moon.

According to Lagrange's calculations, L1, L2, L3 are known as unstable, because a satelite placed there will gradually drift away, and require an adjustment to return to the point. L4 and L5 are known as stable points. We will investigate this assertion in the lab. It has been calculated that there is a minimum ratio of the masses of the two larger bodies that affects the stability of points L4 and L5. This required ratio is approximately 24.96. Supposedly, if the value is lower than 24.96, none of the Lagrangian points are stable. We are most interested in determining the validity of this statement, by changing the mass of the moon, and observing the resultant behavior, both visually and numerically. The points can be visualized in the following figure:

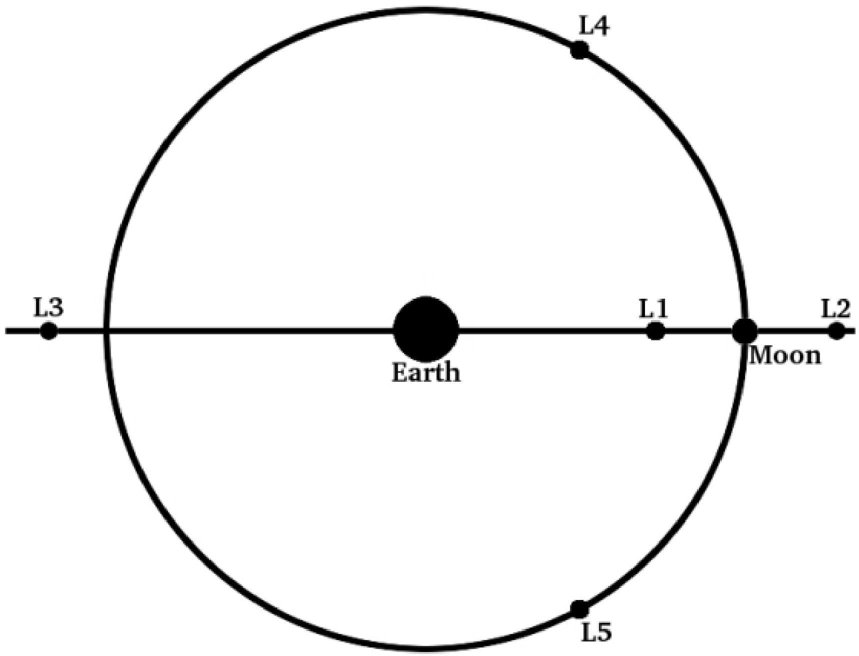

Figure 1: Lagrange Points in the Earth-Moon System

3 Description of Experiment

We started by playing around with the binary star simulation included with the visualpython module. I modified the sample simulation, adding the capability to output the system data, after every iteration, to a file located on the hard disk. This allowed us to import the tens of thousands of data points into a spreadsheet without having to manually key them into the computer. The original simulation had the rate of simulation reduced as to allow for easier viewing of the behavior by the operator. Since we only needed the data points, I essentially removed the rate limitation. This had no effect on the data collected, because it only increased the speed with which we could simulate and collect data points. We obtained standard astronomical values for the earth and moon, which is sumarized at the end of this section. For each iteration of the simulation, the distances between each pair of objects was calculated. The Law of Universal Gravitation was used to find the force between each pair of objects. The force was converted into an impulse, which was applied to the momentum of the bodies. Their positions were updated to reflect the change in momentum, and the simulation proceeded to the next iteration. To investigate the stability ofthe orbits, with the actual astronomical values, we created a satellite at each Lagrange point in turn, calculating the position and initial velocity beforehand. We then recorded the X and Y coordinates for every iteration of the simulation. For each Lagrange point, we recorded 32,000 data points in X and Y. This is equivalent to 31212702.59 seconds, which is just under one year (361.25 days). Each iteration of our simulation was the equivalent of 975.427438 seconds, which is 16.25 minutes. We plotted the data to ascertain any regular harmonic motion, which would be indicitive of a stable orbit.

To determine the effect of the planetary mass ratio, we used the same Lagrange point (L4) each time, but changed the mass of the moon each time. We recorded data for four different ratios. For all of the experiments, $M_e = 5.9736 \cdot 10^{24}$ kg, and the mass of the satellite, $M_s = 10$ kg.

Experiment 1.1 Point L1, $M_m = 7.347673 \cdot 10^{22}$ kg

Experiment 1.2 Point L2, $M_m = 7.347673 \cdot 10^{22}$ kg

Experiment 1.3 Point L3, $M_m = 7.347673 \cdot 10^{22}$ kg

Experiment 1.4 Point L4, $M_m = 7.347673 \cdot 10^{22}$ kg

Experiment 1.5 Point L5, $M_m = 7.347673 \cdot 10^{22}$ kg

Experiment 2.1 Point L4, $M_m = 8.54 \cdot 10^{22}$ kg, ratio = 70

Experiment 2.2 Point L4, $M_m = 1.49 \cdot 10^{23}$ kg, ratio = 40

Experiment 2.3 Point L4, $M_m = 1.99 \cdot 10^{23}$ kg, ratio = 30

Experiment 2.4 Point L4, $M_m = 2.40 \cdot 10^{23}$ kg, ratio = 24.96

Standard Astronomical Data:

Mass of Earth = $5.9736 \cdot 10^{24}$ kg

Radius of Earth = 25512560 m

Mass of Moon = $7.347673 \cdot 10^{22}$ kg

Radius of Moon = 6952400 m

Orbital Radius of Moon = 384400000 m

Orbital Velocity of Moon = $1022 \frac{m}{s}$

4 Data

Please note that the data tables presented only represent a small fraction of our data. We collected 1 data point every 16.25 minutes, for a total of 32,000 data points per table, which would be too many to print here. Also, the values measured and recorded are the values of the x and y coordinates, measured in meters. Time is measured in seconds.

The data in this first section is from the standard planetary masses, for all five Lagrange points. The data is presented first, followed by a data chart and visualization for each trial.

Experiment 1.1: Point L1, Standard Masses		
Time	X	Y
0.00	326378400.00	0.00
975.43	326376105.10	867738.62
1950.85	326371515.28	1735470.99
2926.28	326364630.57	2603190.87
3901.71	326355450.98	3470892.01
4877.14	326343976.58	4338568.17
5852.56	326330207.41	5206213.09
6827.99	326314143.57	6073820.54
7803.42	326295785.16	6941384.26
8778.85	326275132.30	7808898.02

Experiment 1.2: Point L2, Standard Masses		
Time	X	Y
0.00	447677100.00	0.00
975.43	447673873.48	1190234.12
1950.85	447667420.42	2380459.61
2926.28	447657740.83	3570667.83
3901.71	447644834.73	4760850.14
4877.14	447628702.17	5950997.91
5852.56	447609343.22	7141102.51
6827.99	447586757.96	8331155.31
7803.42	447560946.50	9521147.65
8778.85	447531908.96	10711070.92

Experiment 1.3: Point L3, Standard Masses		
Time	X	Y
0.00	-381632000.00	0.00
975.43	-381629243.58	-1014640.75
1950.85	-381623730.73	-2029274.26
2926.28	-381615461.46	-3043893.30
3901.71	-381604435.79	-4058490.62
4877.14	-381590653.77	-5073058.98
5852.56	-381574115.45	-6087591.16
6827.99	-381554820.91	-7102079.91
7803.42	-381532770.24	-8116517.99
8778.85	-381507963.57	-9130898.16

Experiment 1.4: Point L4, Standard Masses		
Time	X	Y
0.00	192200000.00	-332900165.22
975.43	193083740.32	-332386790.65
1950.85	193966136.57	-331871044.96
2926.28	194847182.32	-331352931.59
3901.71	195726871.16	-330832454.03
4877.14	196605196.67	-330309615.75
5852.56	197482152.46	-329784420.28
6827.99	198357732.14	-329256871.13
7803.42	199231929.32	-328726971.85
8778.85	200104737.64	-328194726.00

Experiment 1.5: Point L5, Standard Masses		
Time	X	Y
0.00	192200000.00	332900165.22
975.43	191313584.40	333408790.65
1950.85	190425837.58	333915038.07
2926.28	189536765.97	334418904.03
3901.71	188646376.03	334920385.11
4877.14	187754674.18	335419477.92
5852.56	186861666.90	335916179.08
6827.99	185967360.65	336410485.21
7803.42	185071761.90	336902392.97
8778.85	184174877.13	337391899.03

The data in this section is from the four trials with a variable lunar mass. In each data table and chart, the actual lunar mass and subsequent ratio is displayed. Once again, the data is presented first, followed by charts and visualizations.

Experiment 2.1: Point L4 $M_m = 8.54 \cdot 10^{22}$ kg, ratio = 70		
Time	X	Y
0.00	192200000.00	-332900165.22
975.43	193083743.01	-332386785.99
1950.85	193966144.61	-331871030.96
2926.28	194847198.34	-331352903.54
3901.71	195726897.74	-330832407.19
4877.14	196605236.39	-330309545.38
5852.56	197482207.85	-329784321.58
6827.99	198357805.70	-329256739.30
7803.42	199232023.52	-328726802.06
8778.85	200104854.91	-328194513.40

Experiment 2.2: Point L4 $M_m = 1.49 \cdot 10^{23}$ kg, ratio = 40		
Time	X	Y
0.00	192200000.00	-332900165.22
975.43	193083757.52	-332386760.86
1950.85	193966187.97	-331870955.43
2926.28	194847284.72	-331352752.21
3901.71	195727041.16	-330832154.53
4877.14	196605450.67	-330309165.72
5852.56	197482506.66	-329783789.12
6827.99	198358202.52	-329256028.12
7803.42	199232531.68	-328725886.10
8778.85	200105487.54	-328193366.48

Experiment 2.3: Point L4 $M_m = 1.99 \cdot 10^{23}$ kg, ratio = 30		
Time	X	Y
0.00	192200000.00	-332900165.22
975.43	193083768.81	-332386741.31
1950.85	193966221.70	-331870896.68
2926.28	194847351.92	-331352634.51
3901.71	195727152.71	-330831958.01
4877.14	196605617.34	-330308870.41
5852.56	197482739.07	-329783374.97
6827.99	198358511.18	-329255474.96
7803.42	199232926.93	-328725173.66
8778.85	200105979.61	-328192474.38

Experiment 2.4: Point L4 $M_m = 2.40 \cdot 10^{23}$ kg, ratio = 24.96		
Time	X	Y
0.00	192200000.00	-332900165.22
975.43	193083778.08	-332386725.26
1950.85	193966249.40	-331870848.43
2926.28	194847407.10	-331352537.85
3901.71	195727244.32	-330831796.62
4877.14	196605754.22	-330308627.91
5852.56	197482929.94	-329783034.87
6827.99	198358764.65	-329255020.69
7803.42	199233251.51	-328724588.58
8778.85	200106383.70	-328191741.77

5 Results & Analysis

Based on the data we obtained, and the charts that we created, we decided that the Lagrange points P1, P2, and P3 really are unstable, especially compared to the stability showcased by points L4 and L5. While some people may argue that an orbit that changes to an eliptical orbit closer to the earth is stable, we decideded, given our original definitions of stability, that the satellite needed to stay in the same spot relative to the earth and moon to be considered stable. Therefore, only L4 and L5 are stable Lagrange points. This conclusion is only valid for the actual masses of the earth and moon, as our further experiments showed. While the critical mass-ratio of 24.96 showed a spectactular ejection of the satellite from the system, the simulations when the ratio was 30 and 40 shows that the orbit exhibits instability even at those higher ratios.

6 Error Attribution

As was stated in the lab introduction, neither the general nor restricted three body problem has been solved analytically. Therefore we had to resort to calculating the forces between each pair of the three bodies as described in the experiment description. When using the computer to numerically approximate the velocities and position involved, error must be introduced. Also, we set the simulation to iterate by 16 minutes per iteration, which may seem to be a long time between steps, but the overall timeframe of the simulation (one year), 16 minute chunks are minimal. Ideally, we would be able to handle and work with data with an even higher resolution, perhaps by simulation every minute or every second, however the limitations of our minds to handle such an huge quantity (hundereds of thousands, to millions) of pieces of data was an important factor, as well as the computer's ability to update the chart of points in realtime, when dealing with that many points. As it was, one year worth of iterations produced 32,000 data points, which was the maximum number of data points allowed in the spreadsheet application's charting feature.

7 Estimating Uncertainties

Estimating the uncertainties and error is different for this lab report, because we were unable to actually use a real-world experiment. As mentioned in the previous section, the forces, positions, and velocities involved must be numberically approximated by the computer, and this adds a very small amount of error. However,since the computer is able to use approximately 15 decimal digits through all calculations, the error is negligible. If we were going to try and fit an equation to our data, we would perhaps want to use more points for a higher resolution. However, on the graphs contained in this document, the data points are so closely spaced that they look more like a smooth curve than a collection of unique points. For the purposes of this lab report, the accuracy that we obtained is more than enough.

8 Conclusion

In the course of our numeric simulation, we affirmed many aspects of our prediction. The Lagrange points L1, L2, and L3 are unstable, and L4 and L5 are stable. The only area in which we were slight mistaken was in the critical value of the mass-ratio needed to de-stabilize the orbits of L4 and L5. We had initially thought that the orbits would degenerate almost unnoticibly until the critical ratio was reached, but as our data and graphs showed, the degeneration of the orbits at L4 and L5 started fairly quickly, with very noticible changes occuring with a ratio as high as 40. This is evident in the smaller eliptical orbit that the satellite enters when the ratio is 40, and the large, nearly escaping elipse when the ratio is 30.