# Chaotic Motion of a Double Pendulum

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## 1 Introduction

When studying systems of motion in the real world, a common catchphrase is *'chaotic motion'*. Many people use this term to describe the seeming random behavior of sometimes surprisingly simple objects. An easily created example of this type of system is a double pendulum system, which consists of a pair of rods, the upper rod connected to a fixed point, the lower rod connected only to the upper, that are free to swing. We investigated the chaotic motion of this system, the effect that changes in the initial conditions had in the end result, and methods of mathematically representing the pendulums.

## 2 Predictions

With a normal pendulum, a small change in the initial angle will only change the final (after a set amount of time) angle by approximately the same result. In other words, a linear change in an initial condition will produce a linear change in the resulting motion. With the introduction of a second pendulum, all of our intuition must be disregarded. In the chaotic motion exhibited by the double pendulum system, we predict that a small change will result in two completely different behaviors. We expect this change to become visible almost immediately after starting the system.

## 3 Description of Experiments

Since we did not have access to a double pendulum system, we decided to make modifications to the basic two pendulum simulation that is provided with the visualpython module. We decided that it would make more sense to have the masses and length of each pendulum the same. The two factors that we chose to study were the two angles, $\theta_1$ and $\theta_2$. See Figure 1.

We conducted essentially two experiments, the first exploring the end result of the system when only $\theta_1$ was changing, and the second when only $\theta_2$ was changing.

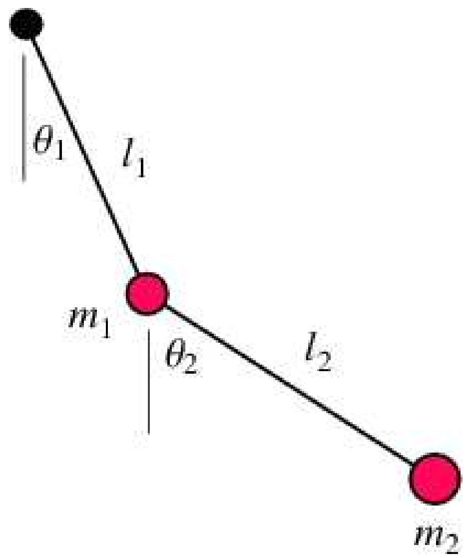


Figure 1: Double Pendulum System

When we finished with those two experiments, we decided we wanted to investigate the results over a longer time period when $\theta_1$ or $\theta_2$ was changed very slightly. We present the results of these longer period experiments after the original experiments

I modified the sample simulation, adding the capability to output the status of the pendulum system, after every iteration, to a file located on the hard disk. This allowed us to import thousands of data points into a spreadsheet without ever having to manually key them into the computer. The original simulation had the rate of simulation reduced as to allow for easier viewing of the behavior by the operator. Since we only needed the data points, I essentially removed the rate limitation. This had no effect on the data collected, because it only increased the speed with which we could simulate and collect data points.

**Experiment 1** $\theta_1 = \{1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3\}$, $\theta_2 = 0$, $0 < t < 15$

**Experiment 2** $\theta_1 = 2$, $\theta_2 = \{0.2, 0.225, 0.25, 0.275, 0.3\}$, $0 < t < 10$

**Experiment 3** $\theta_1 = \{1.7, 1.71\}$, $\theta_2 = 0$, $0 < t < 30$

**Experiment 4** $\theta_1 = 2$, $\theta_2 = \{0.25, 0.2505\}$, $0 < t < 30$

# 4 Data

Please note that the data tables presented only represent a small fraction of our data. We collected 100 data points per second, for a total of 1,000 to 3,000 points per table, which would be too many to print here. Also, the values measured and recorded are the values of $\theta_1$. Angle measurements are in radians, and time measurements are in seconds.

| **Experiment 1:** $\theta_1$ variable, $\theta_2 = 0$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Time | $\theta_1 = 1.7$ | $\theta_1 = 1.8$ | $\theta_1 = 1.9$ | $\theta_1 = 2.0$ | $\theta_1 = 2.1$ | $\theta_1 = 2.2$ | $\theta_1 = 2.3$ |
| 0.01 | 1.70 | 1.80 | 1.90 | 2.00 | 2.10 | 2.20 | 2.30 |
| 0.02 | 1.70 | 1.80 | 1.90 | 2.00 | 2.10 | 2.20 | 2.30 |
| 0.03 | 1.69 | 1.79 | 1.89 | 1.99 | 2.09 | 2.19 | 2.29 |
| 0.04 | 1.68 | 1.78 | 1.88 | 1.99 | 2.09 | 2.19 | 2.29 |
| 0.05 | 1.68 | 1.78 | 1.88 | 1.98 | 2.08 | 2.18 | 2.28 |
| 0.06 | 1.66 | 1.77 | 1.87 | 1.97 | 2.07 | 2.17 | 2.27 |
| 0.07 | 1.65 | 1.75 | 1.85 | 1.96 | 2.06 | 2.16 | 2.26 |
| 0.08 | 1.64 | 1.74 | 1.84 | 1.94 | 2.05 | 2.15 | 2.25 |
| 0.09 | 1.62 | 1.72 | 1.82 | 1.93 | 2.03 | 2.14 | 2.24 |
| 0.10 | 1.60 | 1.70 | 1.81 | 1.91 | 2.02 | 2.12 | 2.23 |

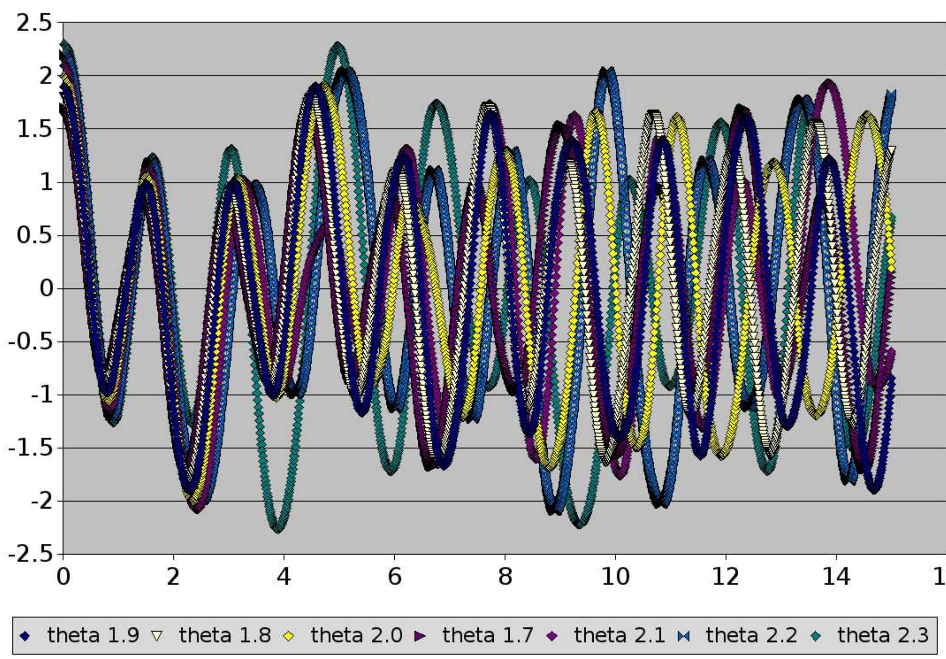


Figure 2: $\theta_1$ Multiple Lines

| Experiment 2: $\theta_1 = 2$, $\theta_2$ variable | | | | | |
|---|---|---|---|---|---|
| Time | $\theta_2 = 0.2$ | $\theta_2 = 0.225$ | $\theta_2 = 0.25$ | $\theta_2 = 0.275$ | $\theta_2 = 0.3$ |
| 9.90 | -0.60 | -0.60 | -1.39 | 1.39 | 1.26 |
| 9.91 | -0.65 | -0.66 | -1.35 | 1.37 | 1.28 |
| 9.92 | -0.69 | -0.73 | -1.30 | 1.36 | 1.29 |
| 9.93 | -0.73 | -0.79 | -1.25 | 1.34 | 1.31 |
| 9.94 | -0.77 | -0.85 | -1.20 | 1.32 | 1.32 |
| 9.95 | -0.81 | -0.91 | -1.14 | 1.30 | 1.33 |
| 9.96 | -0.85 | -0.97 | -1.08 | 1.28 | 1.33 |
| 9.97 | -0.89 | -1.03 | -1.02 | 1.26 | 1.34 |
| 9.98 | -0.92 | -1.09 | -0.96 | 1.23 | 1.34 |
| 9.99 | -0.96 | -1.14 | -0.89 | 1.21 | 1.35 |
| 10.00 | -0.99 | -1.19 | -0.83 | 1.18 | 1.35 |

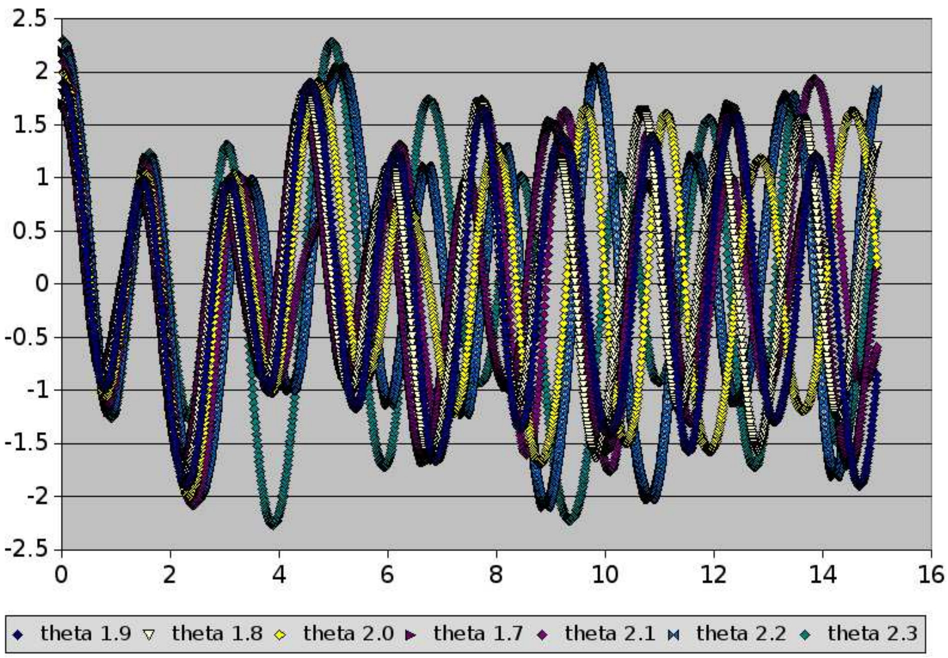


Figure 3: $\theta_2$ Multiple Lines

We wanted to investigate the long-term behavior with small changes in the initial conditions. The results of these two experiments are below.

| **Experiment 3:** $\theta_1 = 2$**,** $\theta_2 = \{1.7, 1.71\}$ | | |
|---|---|---|
| Time | $\theta_2 = 1.7$ | $\theta_2 = 1.71$ |
| 29.91 | 0.00 | -1.20 |
| 29.92 | 0.01 | -1.20 |
| 29.93 | 0.02 | -1.20 |
| 29.94 | 0.03 | -1.20 |
| 29.95 | 0.05 | -1.19 |
| 29.96 | 0.06 | -1.19 |
| 29.97 | 0.07 | -1.18 |
| 29.98 | 0.08 | -1.17 |
| 29.99 | 0.09 | -1.16 |
| 30.00 | 0.10 | -1.15 |

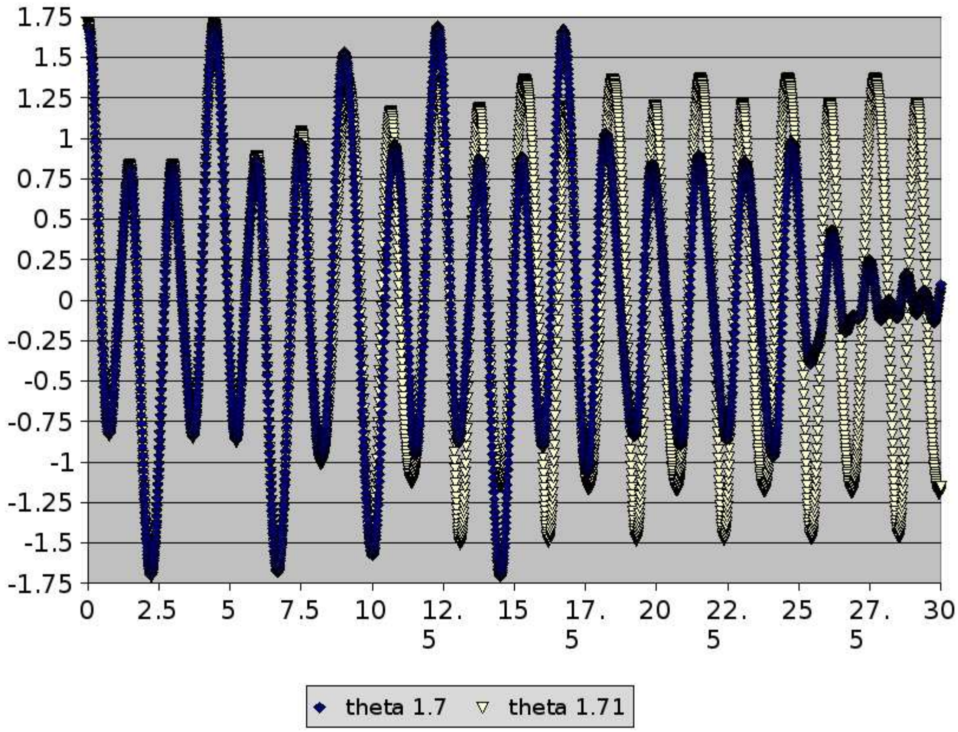


Figure 4: $\theta_1$ Two Lines - Long Duration

| **Experiment 4:** $\theta_1 = \{0.25, 0.2505\}$, $\theta_2 = 0$ | | |
|---|---|---|
| Time | $\theta_1 = 0.25$ | $\theta_1 = 0.2505$ |
| 29.90 | -1.33 | 1.44 |
| 29.91 | -1.31 | 1.46 |
| 29.92 | -1.29 | 1.47 |
| 29.93 | -1.26 | 1.49 |
| 29.94 | -1.23 | 1.50 |
| 29.95 | -1.20 | 1.51 |
| 29.96 | -1.17 | 1.51 |
| 29.97 | -1.13 | 1.52 |
| 29.98 | -1.10 | 1.52 |
| 29.99 | -1.06 | 1.52 |
| 30.00 | -1.02 | 1.52 |

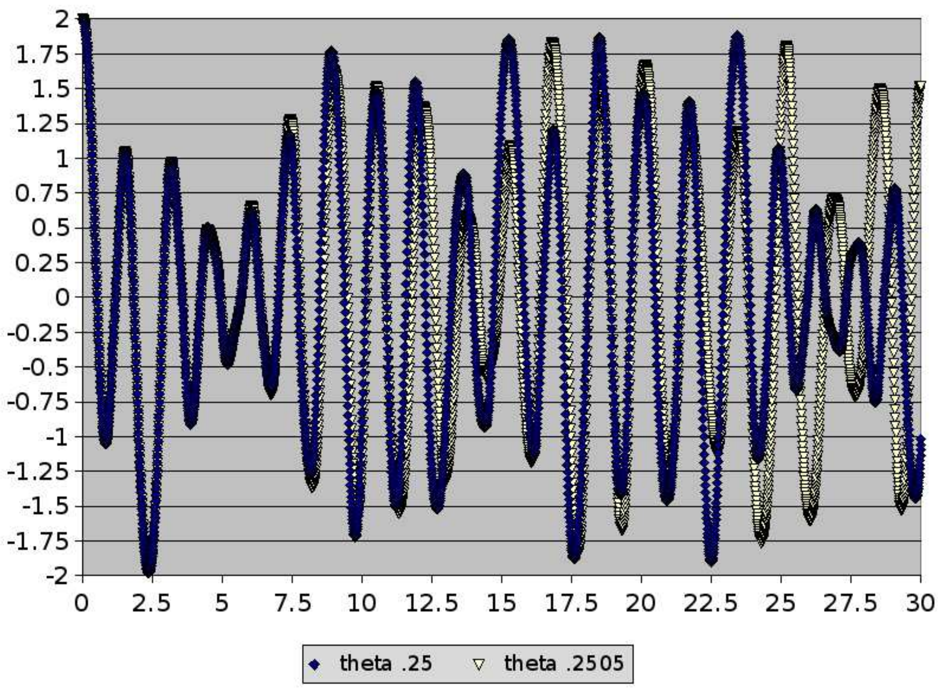


Figure 5: $\theta_2$ Two Lines - Long Duration

# 5 Results & Analysis

Based on the data that we obtained, and the charts that we created using that data, we decided that the double pendulum system really did exhibit chaotic motion. Since part of our assignment was to understand the physics behind our simulation, here is an explanation using *Lagrangian* mathematics.

The Lagrangian is an equation that relates the kinetic and potential energy of a system. It is described as:

$$L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = T_1 + T_2 - V_1 - V_2 \tag{1}$$

Where:

$T_1$ = Kinetic energy of the first rod

$T_2$ = Kinetic energy of the second rod

$V_1$ = Potential energy of the first rod

$V_2$ = Potential energy of the second rod

For the purposes of this explanation:

$$\dot{\theta}_1 = \frac{\partial \theta_1}{\partial t}, \dot{\theta}_2 = \frac{\partial \theta_2}{\partial t}, \ddot{\theta}_1 = \frac{\partial^2 \theta_1}{\partial t}, \ddot{\theta}_2 = \frac{\partial^2 \theta_2}{\partial t}$$

If we decide that the potential energy is equal to zero at the top, then:

$$V_1 = m_1 g y_1, V_2 = m_2 g y_2$$

Using the diagram, a bit of trigonometry, we can write the cartesian coordinates of the centers of mass of the two rods in terms of $a, b, \theta_1, \theta_2$. We can then combine these expressions into our potential energy formulas to produce:

$$V_1 = -m_1 g \frac{a}{2} \cos\theta_1, V_2 = -m_2 g (a \cos\theta_1 + \frac{b}{2} \cos\theta_2)$$

Now we can move on to find the kinetic energy. If we start with the standard for of the kinetic energy of a rigid body:

$$KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

Using a large amount of messy algebra, many time derivatives, and a few trig identities, we can find the kinetic energy of each rod:

$$T_1 = \frac{1}{2} m_1 (\frac{a^2}{4} \dot{\theta_1}^2) + \frac{1}{2} (\frac{1}{12} m_2 a^2) \dot{\theta_1}^2$$

$$T_2 = \frac{1}{2} m_2 (a^2 \dot{\theta_1}^2 + \frac{b^2}{4} \dot{\theta_2}^2 + ab \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) + \frac{1}{2} (\frac{1}{12} m_2 b^2) \dot{\theta_2}^2$$

If we take the kinetic energies and potential energies from both rods, and plug them back into the lagrangian (Equation 1), we get:

$$(\frac{1}{6}m_1a^2 + \frac{1}{2}m_2a^2)\dot{\theta_1}^2 + \frac{1}{6}m_2b^2\dot{\theta_2}^2 + \frac{1}{2}m_2ab\dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2) +$$

$$+(\frac{1}{2}m_1ga + m_2ga)\cos\theta_1 + \frac{1}{2}m_2gb\cos\theta_2$$

Using a bit more algebraic substitution and differientation with respect to time, we get the following relationships:

$$(\frac{1}{3}m_1a^2 + m_2a^2)\ddot{\theta_1} + \frac{1}{2}m_2ab\ddot{\theta_2}\cos(\theta_1 - \theta_2) -$$

$$-\frac{1}{2}m_2ab\dot{\theta_2}^2\sin(\theta_1 - \theta_2) + (\frac{1}{2}m_1ga + m_2ga)\sin\theta = 0 \tag{2}$$

$$\frac{1}{3}m_2b^2\ddot{\theta_2} + \frac{1}{2}m_2ab\ddot{\theta_1}\cos(\theta_1 - \theta_2) -$$

$$-\frac{1}{2}m_2ab\dot{\theta_1}^2\sin(\theta_1 - \theta_2) + \frac{1}{2}m_2gb\sin\theta_2 = 0 \tag{3}$$

Since Equations 2 and 3 are nowhere near linear, it is very difficult to solve the equations symbolically. Using the visualpython modules and the number crunching ability of the computer, it is possible to numerically iterate the equations over a set time interval. This is what we used to obtain all of our data, as well as the demonstration at the end of our presentation.

# 6 Error Attribution

As mentioned in the previous section, the lagrangian equation for the two pendulum system is nearly impossible to solve algebraically except for a select few special cases. When using the computer to numerically approximate the angles and energies involved, error must be introduced. Also, we set the simulation to iterate by thousandths of a second, and we recorded the data every hundredth of a second. Ideally, we would be able to handle and work with data with an even higher resolution, however the limitations of our minds to handle hundereds of thousands of pieces of data was an important factor, as well as the computer's ability to update the chart of points in realtime, when dealing with that many points.

# 7 Estimating Uncertainties

Estimating the uncertainties and error is different for this lab report, because we did not actually use a real-world experiment. As mentioned in the previous section, the lagrangian must be numberically approximated by the computer, and this adds a very small amount of error, but since the computer is able to use approximately 15 decimal digits through all calculations, the error is negligible. If we were going to try and fit a function to our data, we would perhaps want to use more points for a higher resolution. However, on the graphs contained in this document, the data points are so closely spaced that they look more like a smooth curve than a collection of unique points. For the purposes of this lab report, the accuracy that we obtained is more than enough.

# 8 Conclusion

In the course of our numeric simulation, we affirmed many aspects of our prediction. The system does indeed exhibit chaotic motion, and small changed in the initial conditions actually do produce very large and varied end results. The only area in which we were slight mistaken was in the time duration that was required to produce a noticible discrepancy between the two closely started pendulum systems. We had initially thought that the difference would become evident quite quickly, but as our data and graphs showed, the pendulums were relatively close for quite a while. This is evident in the demonstration at the end of the presentation, where we simulate two double-pendulums swinging at the same time, with the same upper pivot position.